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Thermodynamic formulation of a material model for microcracking applied to creep damage

Kari Santaoja*

Department of Applied Mechanics. Aalto University, P.O. Box 14300, FI-00076 Aalto, Finland

Abstract

A Hookean material containing an increasing number of non-interacting microcracks is studied from the aspect of continuum thermodynamics. Rectilinear microcracks in a two-dimensional body are examined. The former model by Basista (2003) is enhanced to describe the response of microcracks to compression. Microcrack densities introduced here enter into the formulation of continuum thermodynamics as internal variables. The strong physical foundation of these internal variables makes them more attractive quantities for damage mechanics than variable damage, the physical background of which is sometimes unclear. The model is coded as an Abaqus VUMAT subroutine and applied to the creep of ice.

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Keywords: Thermodynamics; microcracking; specific Gibbs free energy; Clausius-Duhem inequality; material modelling; Abaqus VUMAT; ice.

1. Introduction

High-speed computers been made it possible to simulate complicated physical processes. Modern computer codes, such as Abaqus, allow users to implement new features into the codes. These two advances provide a tool for introducing detailed material models for engineering purposes. Thus, there is an increasing interest in developing new, more capable material models. Material models have to be verified by comparing their predictions with measured data but this is not enough; validation is also required. In material modelling this is done with the Clausius-Duhem inequality, which is derived from the basic laws and axioms of continuum thermodynamics.

* Corresponding author. Tel.: +358-50-432-6623
E-mail address: kari.santaoja@aalto.fi

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Satisfaction of the Clausius-Duhem inequality ensures that the material model does not violate the basic laws and axioms of nature. Thus, when material models are constructed, continuum thermodynamics and the Clausius-Duhem inequality are obvious tools.

The present work prepares a thermodynamic formulation for a creep damage model where the damage is caused by increasing number of rectilinear non-interacting microcracks in a two-dimensional body. The model is coded as an Abaqus VUMAT subroutine and it is applied to the columnar-grained S2 sea ice.

2. State equations, Clausius-Duhem inequality and the normality rule

Here deformations and rotations are assumed to be small. The (total) strain rate tensor $\dot{\boldsymbol{\varepsilon}}$ is assumed to be separable as follows:

$$\dot{\boldsymbol{\varepsilon}} = \dot{\boldsymbol{\varepsilon}}^{\mathrm{e}} + \dot{\boldsymbol{\varepsilon}}^{\mathrm{d}} + \dot{\boldsymbol{\varepsilon}}^{\mathrm{v}} \qquad \Rightarrow \qquad \boldsymbol{\varepsilon} = \boldsymbol{\varepsilon}^{\mathrm{e}} + \boldsymbol{\varepsilon}^{\mathrm{d}} + \boldsymbol{\varepsilon}^{\mathrm{v}}\,, \tag{1}$$

where $\boldsymbol{\varepsilon}^{\mathrm{e}}$ is the elastic strain tensor for an undamaged material, $\boldsymbol{\varepsilon}^{\mathrm{d}}$ is the damage strain tensor and $\boldsymbol{\varepsilon}^{\mathrm{v}}$ is the creep strain tensor. Eq. $(1)_2$ is obtained by assuming that all the components of the above strain tensors vanish in the initial configuration, i.e. when $t=0$, and after integration of Eq. $(1)_1$ over time t. In terms of the specific Gibbs free energy g, the mechanical part of the state equations for the present material model take the forms

$$\boldsymbol{\varepsilon} - \boldsymbol{\varepsilon}^{\mathrm{v}} = \rho_0 \frac{\partial g(\boldsymbol{\sigma},\boldsymbol{\alpha},Q^r,T)}{\partial \boldsymbol{\sigma}}\,, \qquad \boldsymbol{\beta} = \rho_0 \frac{\partial g(\boldsymbol{\sigma},\boldsymbol{\alpha},Q^r,T)}{\partial \boldsymbol{\alpha}} \quad \text{and} \quad Y^r = \rho_0 \frac{\partial g(\boldsymbol{\sigma},\boldsymbol{\alpha},Q^r,T)}{\partial Q^r}\,. \tag{2}$$

In Eqs (2), ρ_0 is the density in the initial configuration, $\boldsymbol{\sigma}$ is the stress tensor and T is the absolute temperature. The internal variables $\boldsymbol{\alpha}$ and Q^r are defined later. The variables $\boldsymbol{\beta}$ and Y^r are the internal forces associated for the internal variables $\boldsymbol{\alpha}$ and Q^r, respectively. The local forms of the first and second law of thermodynamics give the Clausius-Duhem inequality, viz.

$$\boldsymbol{\sigma}:\dot{\boldsymbol{\varepsilon}}^{\mathrm{v}} + \boldsymbol{\beta}:\dot{\boldsymbol{\alpha}} + \sum_{r=1}^{M} Y^r\,\dot{Q}^r - \frac{\vec{\nabla} T}{T}\cdot\vec{q} \;\geq\; 0\,. \tag{3}$$

The physical meaning of the variable M is defined later. The notation $\vec{\nabla}$ stands for the vector operator "nabla" and $\vec{q}$ is the heat flux vector. Eq. (3) gives the following normality rule for the mechanical variables:

$$\dot{\boldsymbol{\varepsilon}}^{\mathrm{v}} = \rho_0 \frac{\partial \varphi^{\mathrm{c}}_{\mathrm{mech}}(\boldsymbol{\sigma},\boldsymbol{\beta},Y^r;\boldsymbol{\varepsilon},\boldsymbol{\varepsilon}^{\mathrm{v}},\boldsymbol{\alpha},Q^r,T)}{\partial \boldsymbol{\sigma}}\,, \qquad \dot{\boldsymbol{\alpha}} = \rho_0 \frac{\partial \varphi^{\mathrm{c}}_{\mathrm{mech}}(\cdots)}{\partial \boldsymbol{\beta}} \quad \text{and} \quad \dot{Q}^r = \rho_0 \frac{\partial \varphi^{\mathrm{c}}_{\mathrm{mech}}(\cdots)}{\partial Y^r} \tag{4}$$

where $\varphi^{\mathrm{c}}_{\mathrm{mech}}$ is the mechanical part of the specific complementary dissipation function.

3. Specific Gibbs free energy g^{de} for two-dimensional microcracked medium

Based on stress intensity factors K_{I}, K_{II} and K_{III}, Basista (2003) derived an expression of the specific Gibbs free energy g^{de} for a Hookean material with rectilinear non-interacting microcracks in a two-dimensional body. The author enhances this expression and gives it the following appearance for the plane stress:

$$\begin{aligned} g^{\mathrm{de}}(\sigma,Q^r) = {} & \frac{1}{2\,\rho_0}\left[\frac{1}{3(3\,\lambda + 2\,\mu)}\,[\mathbf{1}:\boldsymbol{\sigma}]^2 + \frac{1}{2\,\mu}\,\mathbf{s}:\mathbf{s}\right] \\ & + \frac{\pi\,h}{E}\sum_{r=1}^{M} Q^r\,(a^r)^2 \times \left\{ \vec{n}^r\cdot\boldsymbol{\sigma}\cdot\boldsymbol{\sigma}\cdot\vec{n}^r - [1 - H(\vec{n}^r\cdot\boldsymbol{\sigma}\cdot\vec{n}^r)](\vec{n}^r\cdot\boldsymbol{\sigma}\cdot\vec{n}^r)^2 \right\}, \end{aligned} \tag{5}$$

where λ and μ are the Lamé elastic constants, $\mathbf{1}$ is the second-order identity tensor, $\mathbf{s}$ is the deviatoric stress tensor, h is the thickness of the two-dimensional body and M is the number of microcrack groups. In each group the sizes and orientations of the microcracks are equal. The quantity a^r is the length of the microcrack and the unit normal vector for the microcrack surface is denoted by $\vec{n}^r$, as shown in Fig. 1. The microcrack densities are

$$Q^r = m^r / (\rho_0\, V^{\mathrm{rve}})\,, \tag{6}$$

where m^r is the number of microcracks within the r'th microcrack group and V^{rve} is the volume of the representative volume element. The role of the Heaviside function $H(\vec{n}^r \cdot \boldsymbol{\sigma} \cdot \vec{n}^r)$ is studied later. It is worth noting that the indices of the vectors and tensors in Eq. (5) only take values 1 and 2.

Fig. 1 shows a two-dimensional body with two microcracks. The coordinate system (x_1, x_2, x_3) is the reference coordinate system and the coordinate system (z_1, z_2, z_3) is the microcrack coordinate system showing the orientation of a particular microcrack, which implies that the orientation of the frame (z_1, z_2, z_3) varies from a microcrack to microcrack. The orientation of the unit normal to the microcrack surface $\vec{n}^r$ is not unique, but the opposite direction of $\vec{n}^r$ leads to the same value for the specific Gibbs free energy g^{de}, as can be seen in Eq. (5).

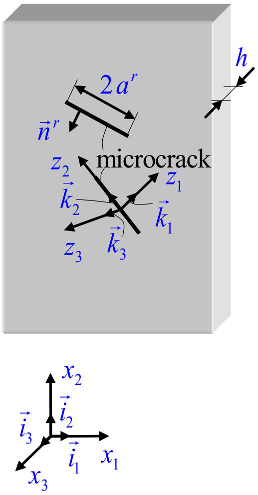

Fig. 1. Microcracks in a two-dimensional body.

Eq. (5) has three major enhancements to the original one proposed by Basista (2003). First, microcracks are collected into groups containing microcracks of the same size and orientation. The introduction of microcrack groups is of course only an approximation given that the size and orientation of microcracks can be randomly distributed. However, microcrack groups make computation faster and the approximation error can be neglected by increasing the number of microcrack groups.

The second enhancement involves the introduction of microcrack densities Q^r, which enter into the formulation of continuum thermodynamics as internal variables. The strong physical foundation of these internal variables makes them more attractive quantities for damage mechanics than variable damage (scalar, vector or tensor), the physical background of which is sometimes unclear.

The third enhancement involves the Heaviside function $H(\vec{n}^r \cdot \boldsymbol{\sigma} \cdot \vec{n}^r)$. Basista (2003) wrote his expression for the specific Gibbs free energy only for tension. The author introduced the Heaviside function $H(\vec{n}^r \cdot \boldsymbol{\sigma} \cdot \vec{n}^r)$ for extending the work by Basista (2003) for compression. The effect of the Heaviside function $H(\vec{n}^r \cdot \boldsymbol{\sigma} \cdot \vec{n}^r)$ is studied next. When expressed in the microcrack frame (z_1, z_2, z_3) the stress tensor $\boldsymbol{\sigma}$ and the normal unit vector to the microcrack surface $\vec{n}^r$ take the following appearances: $\boldsymbol{\sigma} = \overset{\circ}{\sigma}_{ij}\, \vec{k}_i\, \vec{k}_j$ and $\vec{n}^r = \vec{k}_1$, which give

$$\vec{n}^r \cdot \boldsymbol{\sigma} \cdot \boldsymbol{\sigma} \cdot \vec{n}^r = \overset{\circ}{\sigma}_{11}\, \overset{\circ}{\sigma}_{11} + \overset{\circ}{\sigma}_{12}\, \overset{\circ}{\sigma}_{21} \qquad \text{and} \qquad \vec{n}^r \cdot \boldsymbol{\sigma} \cdot \vec{n}^r = \overset{\circ}{\sigma}_{11}\,. \tag{7}$$

The index circle "○" in Eq. (7)1 indicates that the scalar component is expressed in the microcrack coordinate system (z_1, z_2, z_3). Substitution of Eqs (7) into Eq. (5) gives

$$g^{\mathrm{de}}(\sigma, Q^r) \propto \left\{ \left[\overset{\circ}{\sigma}_{11}\, \overset{\circ}{\sigma}_{11} + \overset{\circ}{\sigma}_{12}\, \overset{\circ}{\sigma}_{21} \right] - [1 - H(\overset{\circ}{\sigma}_{11})][\overset{\circ}{\sigma}_{11}]^2 \right\} = H(\overset{\circ}{\sigma}_{11})\, \overset{\circ}{\sigma}_{11}\, \overset{\circ}{\sigma}_{11} + \overset{\circ}{\sigma}_{12}\, \overset{\circ}{\sigma}_{21}\,. \tag{8}$$

As Eq. (8) shows, the role of the Heaviside function $H(\vec{n}^r \cdot \boldsymbol{\sigma} \cdot \vec{n}^r)$ is to neglect the influence of the compressive normal stress $\overset{\circ}{\sigma}_{11}$ on the value of the specific Gibbs free energy g^{de}. This implies that the Heaviside function $H(\vec{n}^r \cdot \boldsymbol{\sigma} \cdot \vec{n}^r)$ prevents the microcrack surfaces from penetrating each other under compression, i.e. when $\overset{\circ}{\sigma}_{11} < 0$, as sketched in Fig. 2.

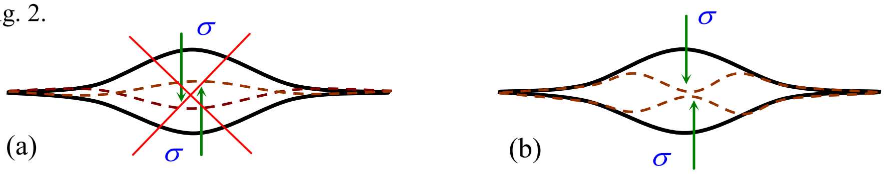

Fig. 2. (a) Surfaces of a microcrack penetrate each other. (b) Penetration is prevented by the Heaviside function $H(\vec{n}^r \cdot \boldsymbol{\sigma} \cdot \vec{n}^r)$.

Eq. (5) is not the only way to apply the Heaviside function $H(\vec{n}^r \cdot \boldsymbol{\sigma} \cdot \vec{n}^r)$ to enhance the expression for the specific Gibbs free energy proposed by Basista (2003). The following enhancement can be introduced:

$${}^{1}g^{\mathrm{de}}(\sigma, Q^r) = \frac{1}{2\,\rho_0} \left[\frac{1}{3(3\,\lambda + 2\,\mu)} [\mathbf{1} : \boldsymbol{\sigma}]^2 + \frac{1}{2\,\mu}\, \mathbf{s} : \mathbf{s} \right] + \frac{\pi\, h}{E} \sum_{r=1}^{M} Q^r\, (a^r)^2 \times \left\{ H(\vec{n}^r \cdot \boldsymbol{\sigma} \cdot \vec{n}^r)\, \vec{n}^r \cdot \boldsymbol{\sigma} \cdot \boldsymbol{\sigma} \cdot \vec{n}^r \right\}. \tag{9}$$

Substitution of Eqs (7) into Eq. (9) gives

$$ {}^{1}g^{\mathrm{de}}(\sigma,Q^{r}) \;\propto\; \left\{ H(\sigma''_{11})\,[\,\sigma''_{11}\,\sigma''_{11} \;+\; \sigma''_{12}\,\sigma''_{21}\,]\right\}. \tag{10} $$

Eq. (10) shows that the first type of the specific Gibbs free energy ${}^{1}g^{\mathrm{de}}$ is for materials in which the forming surfaces of the microcracks are rough, so that even a small compressive stress prevents sliding between the microcrack surfaces. The second type of specific Gibbs free energy g^{de}, Eq. (5), is for materials with no friction between microcrack surfaces. Ice, for example, belongs to this latter type. The specific Gibbs free energy g^{de} is applied below.

4. Strain tensor difference $\boldsymbol{\varepsilon}-\boldsymbol{\varepsilon}^{\mathrm{v}}$ and internal force Y^{r} for a two-dimensional microcracked medium

In the model investigated here the specific Gibbs free energy g^{de} is the only part of the (total) specific Gibbs free energy g which is dependent on the stress tensor $\boldsymbol{\sigma}$. Thus, based on Eqs $(2)_1$ and (5) the following is obtained:

$$ \begin{aligned} \boldsymbol{\varepsilon} \;-\; \boldsymbol{\varepsilon}^{\mathrm{v}} \;=\; \rho_0\,\frac{\partial g^{\mathrm{de}}(\boldsymbol{\sigma},Q^{r})}{\partial \boldsymbol{\sigma}} \;=\; \frac{1}{3(3\,\lambda \;+\; 2\,\mu)}\;\mathbf{1}\,\mathbf{1}:\boldsymbol{\sigma} \;+\; \frac{1}{2\,\mu}\;\mathbf{s} \;+\; \frac{\pi\,h}{E}\sum_{r=1}^{M} Q^{r}\,(a^{r})^{2} \\ \times \left\{ \vec{n}^{r}\boldsymbol{\sigma}\cdot\vec{n}^{r} \;+\; \vec{n}^{r}\cdot\boldsymbol{\sigma}\,\vec{n}^{r} \;-[1-H(\vec{n}^{r}\cdot\boldsymbol{\sigma}\cdot\vec{n}^{r})]\,[\,2\,\vec{n}^{r}\,\vec{n}^{r}\,(\vec{n}^{r}\cdot\boldsymbol{\sigma}\cdot\vec{n}^{r})\,\right\}, \end{aligned} \tag{11} $$

Similarly to the above study, in the model studied here the specific Gibbs free energy g^{de} is the only part of the (total) specific Gibbs free energy g that is dependent on the microcrack densities Q^{r}. Thus, based on Eqs $(2)_3$ and (5) the internal forces Y^{r} are [$r \;=\; 1\ldots M$]

$$ Y^{r} \;=\; \rho_0\,\frac{\partial g(\boldsymbol{\sigma},\boldsymbol{\alpha},Q^{r},T)}{\partial Q^{r}} \;=\; \frac{\pi\,h}{E}\,(a^{r})^{2}\left\{ \vec{n}^{r}\cdot\boldsymbol{\sigma}\cdot\boldsymbol{\sigma}\cdot\vec{n}^{r} \;-[1-H(\vec{n}^{r}\cdot\boldsymbol{\sigma}\cdot\vec{n}^{r})](\vec{n}^{r}\cdot\boldsymbol{\sigma}\cdot\vec{n}^{r})^{2}\right\}. \tag{12} $$

5. Specific Gibbs free energy for creep and the mechanical part of the complementary dissipation function

The specific Gibbs free energy $g(\boldsymbol{\sigma},\boldsymbol{\alpha},Q^{r},T)$ is a sum of three parts as follows:

$$ g(\boldsymbol{\sigma},\boldsymbol{\alpha},Q^{r},T) \;=\; g^{\mathrm{de}}(\boldsymbol{\sigma},Q^{r}) \;+\; g^{\mathrm{v}}(\boldsymbol{\alpha}) \;+\; g^{\mathrm{T}}(T)\,, \qquad \text{where} \qquad g^{\mathrm{v}}(\boldsymbol{\alpha}) \;=\; -\,\frac{m_1}{2\,\rho_0}\;\boldsymbol{\alpha}:\boldsymbol{\alpha}\,. \tag{13} $$

In Eq. $(13)_2$ m_1 is a material parameter. Substitution of Eq. $(13)_1$ into Eq. $(2)_2$ gives

$$ \boldsymbol{\beta} \;=\; \rho_0\,\frac{\partial g(\boldsymbol{\sigma},\boldsymbol{\alpha},Q^{r},T)}{\partial \boldsymbol{\alpha}} \;=\; -m_1\,\boldsymbol{\alpha} \qquad \Rightarrow \qquad \dot{\boldsymbol{\beta}} \;=\; -m_1\,\dot{\boldsymbol{\alpha}}\,. \tag{14} $$

The complementary specific dissipation function related to the viscous deformation $\varphi^{\mathrm{cv}}_{\mathrm{mech}}$ is

$$ \varphi^{\mathrm{cv}}_{\mathrm{mech}}(\boldsymbol{\sigma},\boldsymbol{\beta}) \;=\; \frac{2\,\overset{\circ}{\varepsilon}_{\mathrm{re}}\,\sigma_{\mathrm{re}}}{3\,\rho_0\,(n+1)}\left[\frac{\left\langle J_{\mathrm{vM}}(\boldsymbol{\sigma}-\boldsymbol{\beta}) \;-\; \sigma_{\mathrm{tr}}\right\rangle}{\sigma_{\mathrm{re}}}\right]^{(n+1)}. \tag{15} $$

In Eq. (15) the notations $\overset{\circ}{\varepsilon}_{\mathrm{re}}$, σ_{re}, n σ_{tr} are material parameters. The notation $J_{\mathrm{vM}}(\cdots)$ stands for the von Mises value. The complementary specific dissipation function related to microcracking $\varphi^{\mathrm{cQ}}_{\mathrm{mech}}$ is

$$ \varphi^{\mathrm{cQ}}_{\mathrm{mech}}(Y^{r};\boldsymbol{\varepsilon},\boldsymbol{\varepsilon}^{\mathrm{v}}) \;=\; \frac{\overset{\circ}{Q}_{\mathrm{re}}}{2\,\rho_0}\left\langle\,[\,\vec{n}^{r}\cdot(\boldsymbol{\varepsilon}-\boldsymbol{\varepsilon}^{\mathrm{v}})\cdot\vec{n}^{r}\,]\left\langle\bar{a}\right\rangle \;-\; \varepsilon_{\mathrm{acr}}\right\rangle Y^{r}\,Y^{r}\,, \qquad \text{where} \qquad \bar{a} \;=\; \tfrac{2}{3}\,k^{2}\,[J_{\mathrm{vM}}(\boldsymbol{\varepsilon}^{\mathrm{v}})]\,. \tag{16} $$

In Eqs (16) the notations $\overset{\circ}{Q}_{\mathrm{re}}$, $\varepsilon_{\mathrm{acr}}$ and k^{2} are material parameters. The notation $\langle\cdots\rangle$ indicates Macaulay brackets. The mechanical part of the (total) complementary specific dissipation function $\varphi^{\mathrm{c}}_{\mathrm{mech}}$ is obtained as a sum of the functions $\varphi^{\mathrm{cv}}_{\mathrm{mech}}$ and $\varphi^{\mathrm{cQ}}_{\mathrm{mech}}$. Substitutions of Eqs (15) and (16) into the normality rule given by Eqs $(4)_1$ and $(4)_2$ yields

$$ \dot{\boldsymbol{\varepsilon}}^{\mathrm{v}} \;=\; \rho_0\,\frac{\partial\varphi^{\mathrm{c}}_{\mathrm{mech}}(\cdots)}{\partial\boldsymbol{\sigma}} \;=\; \overset{\circ}{\varepsilon}_{\mathrm{re}}\left[\frac{\left\langle J_{\mathrm{vM}}(\boldsymbol{\sigma}-\boldsymbol{\beta}\,) \;-\; \sigma_{\mathrm{tr}}\right\rangle}{\sigma_{\mathrm{re}}}\right]^{n}\frac{\mathbf{s}-\mathbf{b}}{J_{\mathrm{vM}}(\boldsymbol{\sigma}-\boldsymbol{\beta})} \qquad \text{and} \qquad \dot{\boldsymbol{\alpha}} \;=\; -\,\dot{\boldsymbol{\varepsilon}}^{\mathrm{v}}\,. \tag{17} $$

In Eq. $(17)_1$ the notation $\mathbf{b}$ indicates the deviatoric part of the internal $\boldsymbol{\beta}$. Substitution of the material model given by Eqs (16) into the normality rule given by Eq. $(4)_3$ gives

$$ \dot{Q}^{r} \;=\; \rho_0\,\frac{\partial\varphi^{\mathrm{c}}_{\mathrm{mech}}(\cdots)}{\partial Y^{r}} \;=\; \overset{\circ}{Q}_{\mathrm{re}}\left\langle\,[\,\vec{n}^{r}\cdot(\boldsymbol{\varepsilon}-\boldsymbol{\varepsilon}^{\mathrm{v}})\cdot\vec{n}^{r}\,]\left\langle\bar{a}\right\rangle \;-\; \varepsilon_{\mathrm{acr}}\right\rangle Y^{r}\,. \tag{18} $$

6. Satisfaction of the Clausius-Duhem inequality

Substitution of Eqs (17) and (18) into the Clausius-Duhem inequality given by Eq. (3) gives

$$\overset{\circ}{\varepsilon}_{\rm re}\left[\frac{\left\langle J_{\rm vM}(\boldsymbol{\sigma}-\boldsymbol{\beta})-\sigma_{\rm tr}\right\rangle}{\sigma_{\rm re}}\right]^{n}\frac{\boldsymbol{\sigma}:(\mathbf{s}-\mathbf{b})}{J_{\rm vM}(\boldsymbol{\sigma}-\boldsymbol{\beta})}-\overset{\circ}{\varepsilon}_{\rm re}\left[\frac{\left\langle J_{\rm vM}(\boldsymbol{\sigma}-\boldsymbol{\beta})-\sigma_{\rm tr}\right\rangle}{\sigma_{\rm re}}\right]^{n}\frac{\boldsymbol{\beta}:(\mathbf{s}-\mathbf{b})}{J_{\rm vM}(\boldsymbol{\sigma}-\boldsymbol{\beta})}$$
$$+\sum_{r=1}^{M}\overset{\circ}{Q}_{\rm re}\left\langle [\vec{n}^{r}\cdot(\boldsymbol{\varepsilon}-\boldsymbol{\varepsilon}^{\rm v})\cdot\vec{n}^{r}]\langle\bar{a}\rangle-\varepsilon_{\rm acr}\right\rangle Y^{r}\,Y^{r}\geq 0\,. \tag{19}$$

Since $3/2\,(\boldsymbol{\sigma}-\boldsymbol{\beta}):(\mathbf{s}-\mathbf{b})=3/2\,(\mathbf{s}-\mathbf{b}):(\mathbf{s}-\mathbf{b})=[J_{\rm vM}(\boldsymbol{\sigma}-\boldsymbol{\beta})\,]^{2}$, Eq. (19) reduces to

$$\overset{\circ}{\varepsilon}_{\rm re}\left[\frac{\left\langle J_{\rm vM}(\boldsymbol{\sigma}-\boldsymbol{\beta})-\sigma_{\rm tr}\right\rangle}{\sigma_{\rm re}}\right]^{n}J_{\rm vM}(\boldsymbol{\sigma}-\boldsymbol{\beta})+\sum_{r=1}^{M}\overset{\circ}{Q}_{\rm re}\left\langle [\vec{n}^{r}\cdot(\boldsymbol{\varepsilon}-\boldsymbol{\varepsilon}^{\rm v})\cdot\vec{n}^{r}]\langle\bar{a}\rangle-\varepsilon_{\rm acr}\right\rangle Y^{r}\,Y^{r}\geq 0\,. \tag{20}$$

Eq. (20) shows that the Clausius-Duhen inequality is satisfied.

7. Specific Gibbs free energy $g^{\rm de}$ for penny-shaped microcracks in three-dimensional medium

According to Kachanov (1980) the complementary energy $w^{\rm c}$ for a Hooken material with non-interacting microcracks is given by

$$w^{\rm c}(\boldsymbol{\sigma},\cdots)=\frac{1}{2}\left[\frac{1}{3(3\lambda+2\mu)}[\mathbf{1}:\boldsymbol{\sigma}]^{2}+\frac{1}{2\mu}\,\mathbf{s}:\mathbf{s}\right]+\frac{1}{V^{\rm rve}}\sum_{p=1}^{N}\frac{1}{2}\int_{A^{p}}\vec{b}^{p}\cdot(\boldsymbol{\sigma}\cdot\vec{n}^{p})\,{\rm d}A^{P}\,, \tag{21}$$

where N is the number of microcracks within the representative volume element $V^{\rm rve}$. The area of the p'th microcrack is denoted by A^{p}. The vector $\vec{b}^{p}$ describes the jump between the material points across the microcrack. Here the appearance of the first part of Eq. (21) is changed to be compatible with other equations in this article and the order of the terms within the summation Σ operator.

Santaoja (1988) and (1989) added the Heaviside function $H(\vec{n}^{r}\cdot\boldsymbol{\sigma}\cdot\vec{n}^{r})$ to Eq. (21) for preventing penetration of the microcrack surfaces, as sketched in Figure 2. Here the expression given by Kachanov is further enhanced and the specific Gibbs free energy $g^{\rm de}$ for a three-dimensional Hookean material with non-interacting penny-shaped microcracks is written as follows:

$$g^{\rm de}(\sigma,Q^{r})=\frac{1}{2\rho_{0}}\left[\frac{1}{3(3\lambda+2\mu)}[\mathbf{1}:\boldsymbol{\sigma}]^{2}+\frac{1}{2\mu}\,\mathbf{s}:\mathbf{s}\right]+\frac{16\,(1-\nu^{2})}{3\,(2-\nu)\,E\,\pi^{3/2}}\sum_{r=1}^{M}Q^{r}(A^{r})^{3/2}$$
$$\times\left\{\vec{n}^{r}\cdot\boldsymbol{\sigma}\cdot\boldsymbol{\sigma}\cdot\vec{n}^{r}-[1-H(\vec{n}^{r}\cdot\boldsymbol{\sigma}\cdot\vec{n}^{r})+\tfrac{\nu}{2}H(\vec{n}^{r}\cdot\boldsymbol{\sigma}\cdot\vec{n}^{r})](\vec{n}^{r}\cdot\boldsymbol{\sigma}\cdot\vec{n}^{r})^{2}\right\}. \tag{22}$$

Eq. (22) is a counterpart for Eq. (5) in the sense that compression does not suppress shear deformation.

8. Numerical simulations

The creep-damage material model introduced above was applied to columnar-grained S2 sea ice. The material model was coded as an Abaqus VUMAT subroutine. The numerical values in Table 1 were found to give a good fit with the experimental data obtained in tests carried out at around -10°C.

Table 1. Values of material parameters for columnar-grained S2 ice at around -10°C.

$\rho_0=843\,{\rm kg/m^3}$	$E=9.5\,{\rm GPa}$	$\nu=0.3$	$h=5.10\cdot10^{-2}\,{\rm m}$	$M=15$	$m_1=6.00\,{\rm GPa}$	$k^2=2.00\cdot10^5$
$\overset{\circ}{\varepsilon}_{\rm re}=2.02\cdot10^{-4}\,1/{\rm s}$	$\sigma_{\rm tr}=0.442\,{\rm Pa}$	$\sigma_{\rm re}=2.00\,{\rm MPa}$	$n=1.17$	$\overset{\circ}{Q}_{\rm re}=1.50\cdot10^{12}\,{\rm kg^2\,m/s}$	$\varepsilon_{\rm acr}=1.40\cdot10^{-4}$.	

Fig 3 gives the computed stress-strain relations for tension and for compression at $\dot{\varepsilon}=2\cdot10^{-4}\,1/{\rm s}$ and at $\dot{\varepsilon}=10^{-2}\,1/{\rm s}$. The compressive relation is mirrored so that both curves can be shown in the same quarter of the axes. According to the simulations, the peak values of the stress-strain curves at $\dot{\varepsilon}=2\cdot10^{-4}\,1/{\rm s}$ are $\sigma^{\rm ut}=0.930\,{\rm MPa}$ and $\sigma^{\rm uc}=3.58\,{\rm MPa}$. These values fit well with the values reported by Timco and Frederking (1983) and Timco and

Weeks (2010). The stress-strain curves are cut down at a state where the set of microcracks are assumed to interact or coalesce to form a (macro)crack.

9. Conclusions

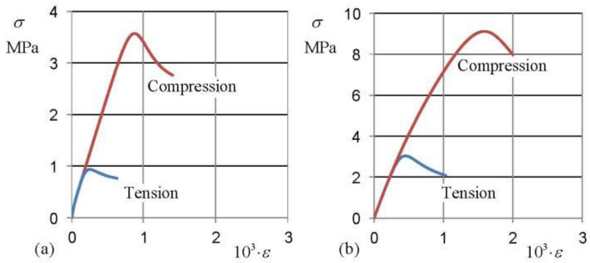

Fig. 3. Stress-strain curves in tension and in compression. (a) at $\dot{\varepsilon} = 2 \cdot 10^{-4}$ 1/s and (b) at $\dot{\varepsilon} = 10^{-2}$ 1/s .

The specific Gibbs free energy g^{de} formulated here for a Hookean material with rectilinear non-interacting microcracks in a two-dimensional domain has three major enhancements to that proposed by Basista (2003). First, the microcracks were collected into groups of same size and orientation, which made the computation much faster. Second, internal variables called microcrack densities Q^r were introduced. They were defined to be $Q^r = m^r / (\rho_0 V^{\mathrm{rve}})$, where m^r is the number of microcracks within the r'th microcrack group, ρ_0 is the material density and V^{rve} is the volume of the representative volume element. Microcrack densities Q^r are much more natural quantities for damage mechanics than the often physically unclear quantity called damage. Furthermore, microcrack densities Q^r enter into the theory as internal variables, being therefore a vital part of continuum thermodynamics. Finally, the model by Basista (2003) was enhanced also to model compression. The damage model described above was used with the introduction of a new creep damage model shown to satisfy the Clausius-Duhem inequality. The model was implemented by writing an Abaqus VUMAT subroutine and it was shown to give reliable results for columnar-grained S2 sea ice.

Acknowledgements

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